A-level

FURTHER MATHEMATICS

7367/2

Paper 2

Mark scheme

June 2023

Version: 1.0 Final

Mark schemes are prepared by the Lead Assessment Writer and considered, together with the relevant questions, by a panel of subject teachers. This mark scheme includes any amendments made at the standardisation events which all associates participate in and is the scheme which was used by them in this examination. The standardisation process ensures that the mark scheme covers the students' responses to questions and that every associate understands and applies it in the same correct way. As preparation for standardisation each associate analyses a number of students' scripts. Alternative answers not already covered by the mark scheme are discussed and legislated for. If, after the standardisation process, associates encounter unusual answers which have not been raised they are required to refer these to the Lead Examiner.

It must be stressed that a mark scheme is a working document, in many cases further developed and expanded on the basis of students' reactions to a particular paper. Assumptions about future mark schemes on the basis of one year's document should be avoided; whilst the guiding principles of assessment remain constant, details will change, depending on the content of a particular examination paper.

Further copies of this mark scheme are available from aqa.org.uk

Mark scheme instructions to examiners

General

The mark scheme for each question shows:

- the marks available for each part of the question
- the total marks available for the question
- marking instructions that indicate when marks should be awarded or withheld including the principle on which each mark is awarded. Information is included to help the examiner make his or her judgement and to delineate what is creditworthy from that not worthy of credit
- a typical solution. This response is one we expect to see frequently. However credit must be given on the basis of the marking instructions.

If a student uses a method which is not explicitly covered by the marking instructions the same principles of marking should be applied. Credit should be given to any valid methods. Examiners should seek advice from their senior examiner if in any doubt.

Key to mark types

M	mark is for method
R	mark is for reasoning
A	mark is dependent on M marks and is for accuracy
B	mark is independent of M marks and is for method and accuracy
E	mark is for explanation
F	follow through from previous incorrect result

Key to mark scheme abbreviations

CAO	correct answer only
CSO	correct solution only
ft	follow through from previous incorrect result
‘their’	indicates that credit can be given from previous incorrect result
AWFW	anything which falls within
AWRT	anything which rounds to
ACF	any correct form
AG	answer given
SC	special case
OE	or equivalent
NMS	no method shown
PI	possibly implied
sf	significant figure(s)
dp	decimal place(s)
ISW	Ignore Subsequent Workings

Examiners should consistently apply the following general marking principles:

No Method Shown

Where the question specifically requires a particular method to be used, we must usually see evidence of use of this method for any marks to be awarded.

Where the answer can be reasonably obtained without showing working and it is very unlikely that the correct answer can be obtained by using an incorrect method, we must award **full marks**. However, the obvious penalty to candidates showing no working is that incorrect answers, however close, earn **no marks**.

Where a question asks the candidate to state or write down a result, no method need be shown for full marks.

Where the permitted calculator has functions which reasonably allow the solution of the question directly, the correct answer without working earns **full marks**, unless it is given to less than the degree of accuracy accepted in the mark scheme, when it gains **no marks**.

Otherwise we require evidence of a correct method for any marks to be awarded.

Diagrams

Diagrams that have working on them should be treated like normal responses. If a diagram has been written on but the correct response is within the answer space, the work within the answer space should be marked. Working on diagrams that contradicts work within the answer space is not to be considered as choice but as working, and is not, therefore, penalised.

Work erased or crossed out

Erased or crossed out work that is still legible and has not been replaced should be marked. Erased or crossed out work that has been replaced can be ignored.

Choice

When a choice of answers and/or methods is given and the student has not clearly indicated which answer they want to be marked, mark positively, awarding marks for all of the student's best attempts. Withhold marks for final accuracy and conclusions if there are conflicting complete answers or when an incorrect solution (or part thereof) is referred to in the final answer.

AS/A-level Maths/Further Maths assessment objectives

AO		Description
AO1	AO1.1a	Select routine procedures
	AO1.1b	Correctly carry out routine procedures
	AO1.2	Accurately recall facts, terminology and definitions
AO2	AO2.1	Construct rigorous mathematical arguments (including proofs)
	AO2.2a	Make deductions
	AO2.2b	Make inferences
	AO2.3	Assess the validity of mathematical arguments
	AO2.4	Explain their reasoning
	AO2.5	Use mathematical language and notation correctly
AO3	AO3.1a	Translate problems in mathematical contexts into mathematical processes
	AO3.1b	Translate problems in non-mathematical contexts into mathematical processes
	AO3.2a	Interpret solutions to problems in their original context
	AO3.2b	Where appropriate, evaluate the accuracy and limitations of solutions to problems
	AO3.3	Translate situations in context into mathematical models
	AO3.4	Use mathematical models
	AO3.5a	Evaluate the outcomes of modelling in context
	AO3.5b	Recognise the limitations of models
	AO3.5c	Where appropriate, explain how to refine models

Q	Marking Instructions	AO	Marks	Typical solution
1	Circles correct answer	1.1b	B1	$2\sinh x$
	Question total		**1**	

Q	Marking Instructions	AO	Marks	Typical solution
2	Circles correct answer	2.2a	B1	$\lim_{x\to\infty}\left(\frac{e^x}{x^5}\right)$
	Question total		**1**	

Q	Marking Instructions	AO	Marks	Typical solution
3	Ticks correct answer	1.1b	B1	$\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 0 & 2 \end{vmatrix}$
	Question total		**1**	

Q	Marking Instructions	AO	Marks	Typical solution
4	Circles correct answer	2.2a	B1	$\{x : x \geq 4\}$
	Question total		**1**	

Q	Marking Instructions	AO	Marks	Typical solution
5	States the correct asymptotes of C_1	1.1b	B1	Josh's method Reflection in $y = x$ C_2 is $\frac{y^2}{16} - \frac{x^2}{9} = 1$ The asymptotes of C_2 are $y = \pm\frac{4}{3}x$ Zoe's method The asymptotes of C_1 are $y = \pm\frac{3}{4}x$ The transformation is a reflection in $y = x$ The asymptotes of C_2 are $y = \pm\frac{4}{3}x$ Both answers are the same, so Zoe is correct.
	States the correct equation of C_2	3.1a	B1	
	States the correct asymptotes of C_2	1.1b	B1	
	Obtains the asymptotes of C_2 by both methods.	3.1a	M1	
	Shows that both methods lead to the same answer and concludes that Zoe is correct.	2.3	R1	
	Question total		**5**	

Q	Marking Instructions	AO	Marks	Typical solution
6(a)	Obtains correct modulus (allow √50) or argument.	1.1b	B1	$5\sqrt{2}e^{i\left(-\frac{3\pi}{4}\right)}$
	Obtains completely correct answer. Allow √50	1.1b	B1	
	Subtotal		**2**	

Q	Marking Instructions	AO	Marks	Typical solution
6(b)	Deduces moduli are all equal.	2.2a	M1	$-\frac{3\pi}{4}+\frac{2\pi}{3}=-\frac{\pi}{12}$ $-\frac{3\pi}{4}+\frac{4\pi}{3}=\frac{7\pi}{12}$
	Adds a multiple of $\frac{2\pi}{3}$ to their argument from part **(a)**.	1.1a	M1	
	Obtains completely correct solution. Allow √50	1.1b	A1	$5\sqrt{2}e^{i\left(-\frac{\pi}{12}\right)}, 5\sqrt{2}e^{i\left(\frac{7\pi}{12}\right)}$
	Subtotal		**3**	

	Question total		**5**	

Q	Marking Instructions	AO	Marks	Typical solution
7	Expresses the sum as the difference of two series.	3.1a	M1	$\sum_{r=11}^{n+1} r^3 = \sum_{r=1}^{n+1} r^3 - \sum_{r=1}^{10} r^3$
	Obtains a correct (unsimplified) expression in terms of n for the sum.	1.1b	A1	$=\frac{1}{4}(n+1)^2(n+2)^2-\frac{1}{4}(10)^2(11)^2$ $=\frac{1}{4}\left\{\left((n+1)(n+2)+110\right)\left((n+1)(n+2)-110\right)\right\}$
	Obtains the required result.	2.1	R1	$=\frac{1}{4}\left\{\left(n^2+3n+112\right)\left(n^2+3n-108\right)\right\}$
	Question total		**3**	

Q	Marking Instructions	AO	Marks	Typical solution
8(a)	Uses the result $(\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ in a statement involving $\mathbf{A}^{-1}$ Use of the notation $\mathbf{A}^{-\mathbf{T}}$ is not acceptable here.	3.1a	M1	$(\mathbf{A}^{\mathrm{T}})(\mathbf{A}^{-1})^{\mathrm{T}} = (\mathbf{A}^{-1}\mathbf{A})^{\mathrm{T}} = \mathbf{I}^{\mathrm{T}} = \mathbf{I}$ $\therefore (\mathbf{A}^{\mathrm{T}})^{-1} = (\mathbf{A}^{-1})^{\mathrm{T}}$
	Uses the fact that the identity matrix is its own transpose. PI	1.1a	M1	
	Completes a reasoned argument to show the required result.	2.1	R1	
	Subtotal		**3**	

Q	Marking Instructions	AO	Marks	Typical solution
8(b)(i)	Obtains $\mathbf{A}^{-1}$ or $\mathbf{A}^{\mathrm{T}}$	1.1a	M1	$\mathbf{A}^{-1} = \dfrac{1}{4k+5}\begin{pmatrix} k & -5 \\ 1 & 4 \end{pmatrix}$
	Obtains $(\mathbf{A}^{-1})^{\mathrm{T}}$ Allow answer with factor outside matrix.	1.1b	A1	$(\mathbf{A}^{-1})^{\mathrm{T}} = \begin{pmatrix} \dfrac{k}{4k+5} & \dfrac{1}{4k+5} \\ \dfrac{-5}{4k+5} & \dfrac{4}{4k+5} \end{pmatrix}$
	Subtotal		**2**	

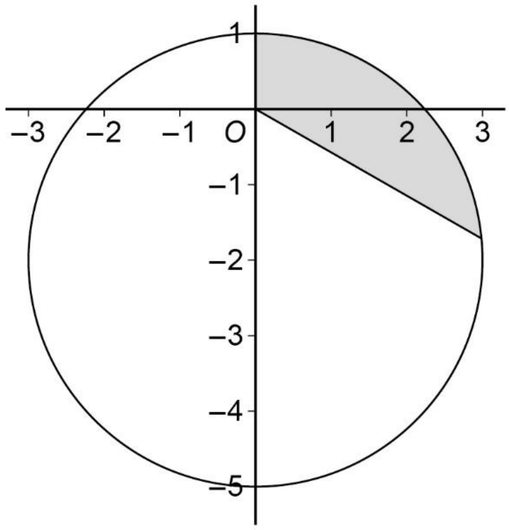
Q	Marking Instructions	AO	Marks	Typical solution
8(b)(ii)	Obtains correct restriction on k FT their det(**A**) from **(b)(i)**.	1.1b	B1F	$k \neq -\dfrac{5}{4}$
	Subtotal		**1**	

	Question total		**6**	

Q	Marking Instructions	AO	Marks	Typical solution
9(a)	Multiplies numerator and denominator by conjugate of denominator.	1.1a	M1	$z=\frac{1+\mathrm{i}}{1-k\mathrm{i}}\times\frac{1+k\mathrm{i}}{1+k\mathrm{i}}=\frac{1-k}{1+k^2}+\mathrm{i}\left(\frac{1+k}{1+k^2}\right)$ Real part $=\frac{1-k}{1+k^2}$ Imaginary part $=\frac{1+k}{1+k^2}$
	Obtains correct real part and correct imaginary part. Condone $\frac{1+k}{1+k^2}\mathrm{i}$	1.1b	A1	
	Subtotal		**2**	

Q	Marking Instructions	AO	Marks	Typical solution
9(b)	Substitutes $k=\sqrt{3}$ into z and finds $\lvert z\rvert$	1.1a	M1	When $k=\sqrt{3}$, $\mathrm{Re}(z)=\frac{1-\sqrt{3}}{4}$ $\lvert z\rvert=\frac{\lvert 1+\mathrm{i}\rvert}{\lvert 1-\sqrt{3}\mathrm{i}\rvert}=\frac{\sqrt{2}}{2}$
	Obtains the correct value for $\lvert z\rvert$	1.1b	A1	
	Obtains $\arg z=\frac{7\pi}{12}$ by a fully correct method.	3.1a	B1	$\arg z=\arg(1+\mathrm{i})-\arg\left(1-\sqrt{3}\mathrm{i}\right)=\frac{\pi}{4}-\left(-\frac{\pi}{3}\right)=\frac{7\pi}{12}$
	Forms an equation of the form $\lvert z\rvert\cos(\arg(z))=\mathrm{Re}(z)$	3.1a	M1	$\frac{\sqrt{2}}{2}\left(\cos\frac{7\pi}{12}\right)=\frac{1-\sqrt{3}}{4}$ $\cos\frac{7\pi}{12}=\frac{\sqrt{2}\left(1-\sqrt{3}\right)}{4}=\frac{\sqrt{2}-\sqrt{6}}{4}$
	Completes a reasoned argument to show the required result.	2.1	R1	
	Subtotal		**5**	

	Question total		**7**	

Q	Marking Instructions	AO	Marks	Typical solution
10(a)	Draws correct arc or circle, intersecting the imaginary axis at 1.	1.1b	B1	
	Draws correct half-line or line at an angle between $-\frac{\pi}{4}$ and 0.	1.1b	B1	
	Shades or clearly labels correct region.	1.1b	B1	
	Subtotal		**3**	

Q	Marking Instructions	AO	Marks	Typical solution
10(b)	Deduces that the maximum value occurs where the half-line $\arg z = -\frac{\pi}{6}$ and the circle intersect. PI	2.2a	M1	Maximum value of $\|z\|$ occurs where circle and half-line intersect. $x = -\sqrt{3}y$ $x^2 + (y+2)^2 = 9$ $3y^2 + y^2 + 4y + 4 - 9 = 0$ $4y^2 + 4y - 5 = 0$ $y = -\frac{1}{2} \pm \frac{\sqrt{6}}{2}$ $y < 0$ so $y = -\frac{1}{2} - \frac{\sqrt{6}}{2}$ $\|y\| = \|z\| \sin\frac{\pi}{6}$ $\|z\| = 2\|y\|$ $\|z\| = 1 + \sqrt{6}$
	Selects a method to form a quadratic equation in x, y or $\|z\|$	3.1a	M1	
	Forms a correct quadratic in x, y or $\|z\|$	2.2a	A1	
	Obtains an expression for the maximum value of $\|z\|$	1.1a	M1	
	Obtains the correct exact value for the maximum value of $\|z\|$ ACF e.g. $\sqrt{7 + 2\sqrt{6}}$	1.1b	A1	
	Subtotal		**5**	

	Question total		**8**	

Q	Marking Instructions	AO	Marks	Typical solution
11(a)	Obtains a direction vector of l_1 PI	1.1a	M1	$\mathbf{r} = \begin{bmatrix} 6 \\ 2 \\ 7 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$
	Obtains a correct Cartesian equation of l_1	1.1b	A1	$x = 6 + 2\lambda, y = 2 + 5\lambda, z = 7$ $\frac{x-6}{2} = \frac{y-2}{5}, z = 7$
	Subtotal		**2**	

Q	Marking Instructions	AO	Marks	Typical solution
11(b)(i)	Obtains correct scalar product of their direction vector of l_1 and the direction vector of l_2	1.1b	B1	Scalar product of direction vectors $= 2\times1 + 5\times1 + 0 = 7$
	Explains that the lines are not perpendicular because this scalar product is non-zero.	2.4	E1	The scalar product is non-zero, so the lines are not perpendicular.
	Subtotal		**2**	

Q	Marking Instructions	AO	Marks	Typical solution
11(b)(ii)	Obtains a vector perpendicular to both lines Or Selects a method to obtain the point of intersection of the two lines.	3.1a	M1	Normal to plane $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 2 & 5 & 0 \end{vmatrix} = \begin{pmatrix} -10 \\ 4 \\ 3 \end{pmatrix}$ Equation of plane is $\mathbf{r} \bullet \begin{pmatrix} -10 \\ 4 \\ 3 \end{pmatrix} = d$ $d = \begin{pmatrix} -10 \\ 4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 6 \\ 2 \\ 7 \end{pmatrix} = -31$ $\begin{pmatrix} 8 \\ 9 \\ c \end{pmatrix} \bullet \begin{pmatrix} -10 \\ 4 \\ 3 \end{pmatrix} = -31$ $-80 + 36 + 3c = -31$ $c = \frac{13}{3}$
	Uses scalar product of their normal vector and the position vector of a point on l_1 or l_2 to obtain constant term in equation of plane. PI Or Forms two simultaneous equations in λ and μ only.	1.1a	M1	
	Obtains correct equation of plane. Or Obtains correct simultaneous equations.	1.1b	A1	
	Forms and solves equation in c using their equation of the plane or the solutions to their simultaneous equations.	1.1a	M1	
	Obtains correct value of c	1.1b	A1	
	Subtotal		**5**	

	Question total		**9**	

Q	Marking Instructions	AO	Marks	Typical Solution
12	Shows that $\mathrm{f}(n)$ is divisible by 19 for $n=1$	1.1b	B1	Let $n=1$; then the formula gives $\mathrm{f}(1)=3^4+2^7=209=11\times19$
	States the assumption that $\mathrm{f}(n)$ is divisible by 19 for $n=k$	2.4	M1	so the result is true for $n=1$
	Expresses $\mathrm{f}(k+1)$ in terms of k	3.1a	M1	Assume the result is true for $n=k$, so
	Expresses $\mathrm{f}(k+1)$ or $\mathrm{f}(k+1)-\mathrm{f}(k)$ in the form $a\left(3^{3k+1}\right)+b\left(2^{3k+4}\right)$	3.1a	M1	$\mathrm{f}(k)=3^{3k+1}+2^{3k+4}=19m\ (m\in\mathbb{Z})$ Then
	Completes reasoned working to correctly deduce that $\mathrm{f}(k+1)$ is divisible by 19	2.2a	R1	$\mathrm{f}(k+1)=3^{3k+4}+2^{3k+7}$ $=27\left(3^{3k+1}\right)+8\left(2^{3k+4}\right)$ $=19\left(3^{3k+1}\right)+8\left(3^{3k+1}\right)+8\left(2^{3k+4}\right)$
	Concludes a reasoned argument by stating that $\mathrm{f}(n)$ is divisible by 19 for $n=1$; if true for $n=k$, then it's also true for $n=k+1$ and hence by induction $\mathrm{f}(n)$ is divisible by 19 for $n\geq1$	2.1	R1	$=19\left(3^{3k+1}\right)+8\mathrm{f}(k)$ $=19\left(3^{3k+1}+8m\right)$ and the result also holds for $n=k+1$ $\mathrm{f}(n)$ is divisible by 19 for $n=1$; if true for $n=k$, then it's also true for $n=k+1$ and hence by induction $\mathrm{f}(n)$ is divisible by 19 for $n\geq1$
	Question total		**6**	

Q	Marking Instructions	AO	Marks	Typical solution
13(a)	Obtains correct sum of roots.	1.1b	B1	$\alpha+\beta=5$ $\alpha\beta=8$
	Obtains correct product of roots.	1.1b	B1	
	Subtotal		**2**	

Q	Marking Instructions	AO	Marks	Typical solution
13(b)	Expresses $\alpha^2+\beta^2$ in terms of $\alpha+\beta$ and $\alpha\beta$	1.1a	M1	$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$ = 25 - 16 = 9 $\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2\alpha^2\beta^2$ $= 9^2 - 2 \times 8^2$ $=-47$
	Obtains correct value of $\alpha^2+\beta^2$	1.1b	A1	
	Expresses $\alpha^4+\beta^4$ in terms of sums and/or products of α, β, α^2, β^2	2.2a	M1	
	Completes a reasoned argument to obtain the required result. CSO	2.1	R1	
	Subtotal		**4**	

Q	Marking Instructions	AO	Marks	Typical solution
13(c)	Expresses the sum of roots of the new equation in terms of sums and/or products of α and β or α^2 and β^2	3.1a	M1	Sum of roots $= \alpha^3 + \beta^3 + \alpha + \beta$ $= (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) + \alpha + \beta$ = 125 - 3 × 8 × 5 + 5 = 10 Product of roots $= (\alpha^3 + \beta)(\beta^3 + \alpha)$ $= \alpha^3\beta^3 + \alpha^4 + \beta^4 + \alpha\beta$ $= 8^3 - 47 + 8$ = 473 $z^2 - 10z + 473 = 0$
	Obtains correct sum of roots.	1.1b	A1	
	Expresses the product of roots of the new equation as $= \alpha^3\beta^3 + \alpha^4 + \beta^4 + \alpha\beta$	1.1a	M1	
	Obtains correct product of roots.	1.1b	A1	
	Deduces a correct quadratic equation with integer coefficients. Allow any variable.	2.2a	A1	
	Subtotal		**5**	

	Question total		**11**	

Q	Marking Instructions	AO	Marks	Typical solution
14(a)	Completes the square for denominator. Or Sets $\mathrm{f}(x)=k$ and forms a quadratic equation in x	3.1a	M1	$\mathrm{f}(x)=\dfrac{1}{4\left(x^2+4x+\frac{19}{4}\right)}$ $=\dfrac{1}{4\left((x+2)^2+\frac{3}{4}\right)}$ f is maximum when the denominator is minimum, that is when $x=-2$ and $y=\dfrac{1}{4\left(\frac{3}{4}\right)}=\dfrac{1}{3}$ So the graph of $y=\mathrm{f}(x)$ has a stationary point at $\left(-2,\frac{1}{3}\right)$
	Explains that f has a stationary point when $x=-2$ Or Equates the discriminant of the quadratic equation to 0 and solves for k	2.4	E1	
	Completes a reasoned argument, without using calculus, to show that the stationary point is at $\left(-2,\dfrac{1}{3}\right)$ to obtain the required result.	2.1	R1	
	Subtotal		**3**	

Q	Marking Instructions	AO	Marks	Typical solution
14(b)	Expresses the denominator of the integrand in completed square form.	3.1a	M1	$\int_{-2}^{-\frac{1}{2}} \mathrm{f}(x)\mathrm{d}x = \frac{1}{4}\int_{-2}^{-\frac{1}{2}} \frac{1}{(x+2)^2+\frac{3}{4}}\mathrm{d}x$ $= \frac{1}{4}\times\frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2(x+2)}{\sqrt{3}}\right)\right]_{-2}^{-\frac{1}{2}}$ $= \frac{1}{2\sqrt{3}}\left(\tan^{-1}\sqrt{3}-\tan^{-1}0\right)$ $= \frac{1}{2\sqrt{3}}\left(\frac{\pi}{3}-0\right) = \frac{\pi}{6\sqrt{3}} = \frac{\pi\sqrt{3}}{18}$
	Uses inverse tan to integrate their integrand of the form $\frac{1}{(x+k)^2+a^2}$ Or Makes a correct substitution	3.1a	M1	
	Integrates to obtain $A\tan^{-1}\frac{2(x+2)}{\sqrt{3}}$	1.1b	A1	
	Substitutes the upper limit correctly into their integrated expression which includes $\tan^{-1}$	1.1a	M1	
	Completes a reasoned argument to obtain the required result. Must see substitution of -2 in the integrated expression.	2.1	R1	
	Subtotal		**5**	

Q	Marking Instructions	AO	Marks	Typical solution
14(c)	Replaces ∞ by a letter (N) and considers $\lim\limits_{N\to\infty}$ in the integral or integrated expression.	3.1a	E1	$\int_{-2}^{\infty} \mathrm{f}(x)\mathrm{d}x = \frac{1}{2\sqrt{3}}\lim\limits_{N\to\infty}\left[\tan^{-1}\left(\frac{2(x+2)}{\sqrt{3}}\right)\right]_{-2}^{N}$ $= \frac{1}{2\sqrt{3}}\left(\frac{\pi}{2}-0\right)$ $= \frac{\pi\sqrt{3}}{12}$
	Obtains the correct exact value. ACF	2.2a	B1	
	Subtotal		**2**	

	Question total		**10**	

Q	Marking Instructions	AO	Marks	Typical solution
15(a)	Uses de Moivre's theorem.	1.1a	M1	By de Moivre's theorem, $z^n = \cos n\theta + \mathrm{i}\sin n\theta$ $z^{-n} = \cos(-n\theta) + \mathrm{i}\sin(-n\theta)$ $= \cos(n\theta) - \mathrm{i}\sin(n\theta)$ $z^n - z^{-n} = 2\mathrm{i}\sin n\theta$
	Completes a reasoned argument to obtain the required result. Must see $\cos(-n\theta) + \mathrm{i}\sin(-n\theta)$ and $z^n - z^{-n}$	2.1	R1	
	Subtotal		**2**	

Q	Marking Instructions	AO	Marks	Typical solution
15(b)	Uses part **(a)** to express at least three terms of S in terms of z	3.1a	M1	$2\mathrm{i}S = 2\mathrm{i}\sin\theta + 2\mathrm{i}\sin 3\theta + \ldots + 2\mathrm{i}\sin(2n-1)\theta$ $= z - z^{-1} + z^3 - z^{-3} + \ldots + z^{2n-1} - z^{-(2n-1)}$ $= z + z^3 + \ldots + z^{2n-1} - \left(z^{-1} + z^{-3} + \ldots + z^{-(2n-1)}\right)$ $S = \frac{1}{2\mathrm{i}}\left(z + z^3 + \ldots + z^{2n-1}\right) - \frac{1}{2\mathrm{i}}\left(z^{-1} + z^{-3} + \ldots + z^{-(2n-1)}\right)$
	Expresses S or $2\mathrm{i}S$ as the difference of two series.	1.1a	M1	
	Completes a reasoned argument to obtain the required result.	2.1	R1	
	Subtotal		**3**	

Q	Marking Instructions	AO	Marks	Typical solution
15(c)	Obtains expressions for the sums of their geometric series.	3.1a	M1	$2\mathrm{i}S = \dfrac{z\left(1-z^{2n}\right)}{\left(1-z^2\right)} - \dfrac{z^{-1}\left(1-z^{-2n}\right)}{\left(1-z^{-2}\right)}$ $= \dfrac{z^{2n}-1}{z-z^{-1}} - \dfrac{1-z^{-2n}}{z-z^{-1}} = \dfrac{z^{2n}+z^{-2n}-2}{z-z^{-1}}$ $= \dfrac{\left(z^n-z^{-n}\right)^2}{2\mathrm{i}\sin\theta} = \dfrac{\left(2\mathrm{i}\sin n\theta\right)^2}{2\mathrm{i}\sin\theta} = -\dfrac{2\sin^2 n\theta}{\mathrm{i}\sin\theta}$ $-2S = -\dfrac{2\sin^2 n\theta}{\sin\theta}$ $S = \dfrac{\sin^2 n\theta}{\sin\theta}$
	Obtains fully correct expressions for the sums of G_1 and G_2	1.1b	A1	
	Rearranges to obtain $z-z^{-1}$ in the denominator of any fraction.	3.1a	B1	
	Obtains $z^{2n}+z^{-2n}-2$ in the numerator of their single fraction.	3.1a	B1	
	Completes a reasoned argument to obtain the required result.	2.1	R1	
	Subtotal		**5**	

	Question total		**10**	

<table>
<tr><th>Q</th><th>Marking Instructions</th><th>AO</th><th>Marks</th><th>Typical Solution</th></tr>
<tr><td rowspan="6">16(a)(i)</td><td>Forms an equation of motion with at least three terms correct.
Accept a and/or v.</td><td>3.3</td><td>M1</td><td rowspan="6">$m\ddot{x} = mg - kx - R\dot{x}$
$m\ddot{x} + R\dot{x} + kx = mg$
CF:
$m\lambda^2 + R\lambda + k = 0$
$\lambda = \frac{-R \pm \sqrt{R^2 - 4km}}{2m} = -\frac{R}{2m} \pm \mathrm{i}\left(\frac{\sqrt{4km - R^2}}{2m}\right)$
CF: $x = \mathrm{e}^{-\frac{Rt}{2m}}\left(A\cos\left(\frac{\sqrt{4km - R^2}}{2m}\right)t + B\sin\left(\frac{\sqrt{4km - R^2}}{2m}\right)t\right)$
PI:
$x = p,\ \dot{x} = 0,\ \ddot{x} = 0 \Rightarrow kp = mg \Rightarrow p = \frac{mg}{k}$
So
$x = \mathrm{e}^{-\frac{Rt}{2m}}\left(A\cos\left(\frac{\sqrt{4km - R^2}}{2m}\right)t + B\sin\left(\frac{\sqrt{4km - R^2}}{2m}\right)t\right) + \frac{mg}{k}$</td></tr>
<tr><td>Obtains fully correct differential equation.</td><td>1.1b</td><td>A1</td></tr>
<tr><td>Obtains solutions of their auxiliary equation.</td><td>1.1a</td><td>M1</td></tr>
<tr><td>Obtains a complementary function consistent with their solutions to their auxiliary equation.
Must justify the choice of complementary function either from the roots of their auxiliary equation or by a clear explanation.</td><td>1.1b</td><td>A1</td></tr>
<tr><td>Uses a correct method to obtain the correct particular integral.</td><td>1.1b</td><td>B1</td></tr>
<tr><td>Completes a fully correct reasoned argument to obtain the required result.</td><td>2.1</td><td>R1</td></tr>
<tr><td></td><td>Subtotal</td><td></td><td>6</td><td></td></tr>
</table>

Q	Marking Instructions	AO	Marks	Typical Solution
16(a)(ii)	Correctly substitutes all relevant values into the general equation.	3.1b	B1	$x = e^{-0.16t}\left(A\cos(0.48t) + B\sin(0.48t)\right) + 38.3$ When $t = 0$, $x = 0$
	Substitutes $t = 0$ and $x = 0$ into the general equation, either with or without other values substituted.	3.4	M1	$0 = A + 38.3$ $A = -38.3$
	Obtains the correct value for A Must have -38.3 or better.	1.1b	A1	$x = e^{-0.16t}\left(-38.3\cos(0.48t) + B\sin(0.48t)\right) + 38.3$ $\dot{x} = -0.16e^{-0.16t}\left(-38.3\cos(0.48t) + B\sin(0.48t)\right)$ $+e^{-0.16t}\left(18.4\sin(0.48t) + 0.48B\cos(0.48t)\right)$
	Uses the product rule to differentiate the expression for displacement.	1.1a	M1	When $t = 0$, $\dot{x} = 14$ $14 = -0.16(-38.3) + 0.48B$ $B = 16.4$
	Substitutes $t = 0$ and $v = 14$ into their equation for v, either with or without other values substituted.	3.4	M1	To the nearest integer, $A = -38$ and $B = 16$
	Completes a fully correct argument to show that to the nearest integer, $A = -38$ and $B = 16$	1.1b	A1	
	Subtotal		**6**	

Q	Marking Instructions	AO	Marks	Typical solution
16(b)	Deduces that $R = 0$ in the equation from part **(a)(i)** OR Obtains a correct solution to the differential equation formed from an equation of motion using $R = 0$	2.2a	B1	Setting $R = 0$ $x = \left(A\cos\left(\frac{\sqrt{4km}}{2m}\right)t + B\sin\left(\frac{\sqrt{4km}}{2m}\right)t\right) + \frac{mg}{k}$ $x = \left(A\cos(0.506t) + B\sin(0.506t)\right) + 38.3$ When $t = 0$, $x = 0 \Rightarrow A = -38.3$
	Differentiates displacement	1.1a	M1	$x = \left(-38.3\cos(0.506t) + B\sin(0.506t)\right) + 38.3$
	Substitutes $t = 0$, $x = 0$ and $v = 14$ into their equations for x and v	3.4	M1	$\dot{x} = 19.4\sin(0.506t) + 0.506B\cos(0.506t)$ When $t = 0$, $\dot{x} = 14$
	Obtains a completely correct expression for x, with values to 2 significant figures or better.	1.1b	A1	$B = \frac{14}{0.506} = 27.7$ $x = \left(-38\cos(0.51t) + 28\sin(0.51t)\right) + 38$
	Subtotal		**4**	

	Question total		**16**	

	Paper total		**100**	